FOXBY EQUIVALENCE, LOCAL DUALITY AND GORENSTEIN HOMOLOGICAL DIMENSIONS

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Abstract. Let $(R,\mathfrak{m})$ be a local ring and $(-)^\vee$ denote the Matlis duality functor. We investigate the relationship between Foxby equivalence and local duality through generalized local cohomology modules. Assume that R possesses a normalized dualizing complex D and X and Y are two homologically bounded complexes of R-modules with finitely generated homology modules. We present several duality results for $\mathfrak{m}$-section complex $\mathbf{R}\Gamma_\mathfrak{m}(\mathbf{R}\operatorname{Hom}_R(X,Y))$. In particular, if G-dimension of X and injective dimension of Y are finite, then we show that

$$\mathbf{R}\Gamma_\mathfrak{m}(\mathbf{R}\operatorname{Hom}_R(X,Y)) \simeq (\mathbf{R}\operatorname{Hom}_R(Y, D\otimes^{\mathbf{L}}_R X))^\vee.$$

We deduce several applications of these duality results. In particular, we establish Grothendieck's non-vanishing Theorem in the context of generalized local cohomology modules.

1. Introduction

Let $(R,\mathfrak{m})$ be a Cohen-Macaulay local ring with a canonical module ω and $\mathcal{P}$ (resp. $\mathcal{I}$) denote the full subcategory of finitely generated R-modules of finite projective (resp. injective) dimension. By virtue of [Sh, Theorem 2.9], there is the following equivalence of categories

$$\begin{tikzcd} \mathcal{P} \arrow[r, shift left, "\omega\otimes_R -"] & \mathcal{I} \arrow[l, shift left, "\operatorname{Hom}_R(\omega,-)"] \end{tikzcd} .$$

Let M and N be two finitely generated R-modules and i a non-negative integer. Denote the Matlis duality functor $\operatorname{Hom}_R(-,E(R/\mathfrak{m}))$ by $(-)^\vee$. If M has finite projective dimension, then by Suzuki's Duality Theorem [Su, Theorem 3.5], there is a natural isomorphism $H^i_\mathfrak{m}(M,N)\cong \operatorname{Ext}^{\dim R-i}_R(N,\omega\otimes_R M)^\vee$. Also, if N has finite injective dimension, then the Herzog-Zamani Duality Theorem [HZ, Theorem 2.1 b)] asserts that $H^i_\mathfrak{m}(M,N)\cong \operatorname{Ext}^{\dim R-i}_R(\operatorname{Hom}_R(\omega,N),M)^\vee$. (These results can be considered as variants of the Local Duality Theorem [BS, 11.2.8] in the context of generalized local cohomology modules.) Hence the equivalence between two subcategories $\mathcal{P}$ and $\mathcal{I}$ can be connected to local duality through generalized local cohomology modules.

Now, assume that $(R,\mathfrak{m})$ is a local ring with a normalized dualizing complex D. By [CFrH, Theorem 4.1 and Prposition 3.8 b)] (resp. [CFrH, Theorem 4.4]) Auslander category $\mathcal{A}^f(R)$ (resp. $\mathcal{B}^f(R)$) consists exactly of all homologically bounded complexes of R-modules with finitely generated homology modules of finite G-dimension (resp. Gorenstein injective dimension). By Foxby equivalence, there is an equivalence

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of categories

$$\mathcal{A}^f(R) \underset{\mathbf{R}\operatorname{Hom}_R(D,-)}{\overset{D\otimes^{\mathbf{L}}_R -}{\rightleftarrows}} \mathcal{B}^f(R) \ ;$$

see e.g. [C, Theorem 3.3.2 a), b), e) and f)]. Foxby equivalence between the two categories $\mathcal{A}^f(R)$ and $\mathcal{B}^f(R)$ is a natural generalization of the above mentioned equivalence between the subcategories $\mathcal{P}$ and $\mathcal{I}$. In view of what we saw in the first paragraph, it is natural to ask whether Foxby equivalence can also be connected to local duality through generalized local cohomology modules. Assume that X and Y are two homologically bounded complexes of R-modules with finitely generated homology modules and let i be an integer. The following natural questions arise:

Question 1.1. Suppose that G-dimension of X is finite. Is $H^i_{\mathfrak{m}}(X,Y) \cong \operatorname{Ext}^{-i}_R(Y, D\otimes^{\mathbf{L}}_R X)^\vee$?

Question 1.2. Suppose that Gorenstein injective dimension of Y is finite. Is

$$H^i_{\mathfrak{m}}(X,Y) \cong \operatorname{Ext}^{-i}_R(\mathbf{R}\operatorname{Hom}_R(D,Y),X)^\vee?$$

Our main aim in this paper is to answer these questions. Example 3.6 below shows that the answers of these questions are negative in general, but by adding some extra assumptions on the complexes X and Y, we can deduce our desired natural isomorphisms. Consider the following assumptions:

a) Projective dimension of X is finite.
b) Projective dimension of Y is finite.
b') Both G-dimension of X and projective dimension of Y are finite.
c) Both G-dimension of X and injective dimension of Y are finite.
d) Injective dimension of Y is finite.
e) Injective dimension of X is finite.
e') Both Gorenstein injective dimension of Y and injective dimension of X are finite.
f) Both Gorenstein injective dimension of Y and projective dimension of X are finite.

We show that each of a), b) and c) implies the natural isomorphism

$$\mathbf{R}\Gamma_{\mathfrak{m}}(\mathbf{R}\operatorname{Hom}_R(X,Y)) \simeq (\mathbf{R}\operatorname{Hom}_R(Y, D\otimes^{\mathbf{L}}_R X))^\vee,$$

and each of d), e) and f) implies the natural isomorphism

$$\mathbf{R}\Gamma_{\mathfrak{m}}(\mathbf{R}\operatorname{Hom}_R(X,Y)) \simeq (\mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_R(D,Y),X))^\vee.$$

These immediately yield our desired isomorphisms $H^i_{\mathfrak{m}}(X,Y) \cong \operatorname{Ext}^{-i}_R(Y, D\otimes^{\mathbf{L}}_R X)^\vee$ and $H^i_{\mathfrak{m}}(X,Y) \cong \operatorname{Ext}^{-i}_R(\mathbf{R}\operatorname{Hom}_R(D,Y),X)^\vee$, respectively. These duality results are far reaching generalizations of Suzuki's Duality Theorem and the Herzog-Zamani Duality Theorem.

We present some applications of the above duality results. First of all, we improve the main results of [HZ]; see Propositions 4.3 and 4.4 below. Then we establish an analogue of Grothendieck's non-vanishing Theorem in the context of generalized local cohomology modules. Let $(R,\mathfrak{m})$ be a local ring and M,N two finitely generated R-modules such that $X := M, Y := N$ satisfy one of the above assumptions a), b'), d), and e'). When R is Cohen-Macaulay, we show that

$$\operatorname{cd}_{\mathfrak{m}}(M,N) = \dim R - \operatorname{depth}(\operatorname{Ann}_R N, M).$$

Finally, we give a partial generalization of the Intersection inequality; see Proposition 4.7 below.

2. Prerequisites

Throughout this paper, R is a commutative Noetherian ring with nonzero identity. The $\mathfrak{m}$-adic completion of an R-module M over a local ring $(R,\mathfrak{m})$ will be denoted by $\widehat{M}$.

(2.1) Hyperhomology. We will work within $\mathcal{D}(R)$, the derived category of R-modules. The objects in $\mathcal{D}(R)$ are complexes of R-modules and symbol $\simeq$ denotes isomorphisms in this category. For a complex

$$X = \cdots \longrightarrow X_{n+1} \xrightarrow{\partial_{n+1}^X} X_n \xrightarrow{\partial_n^X} X_{n-1} \longrightarrow \cdots$$

in $\mathcal{D}(R)$, its supremum and infimum are defined, respectively, by $\sup X := \sup\{i \in \mathbb{Z} | H_i(X) \neq 0\}$ and $\inf X := \inf\{i \in \mathbb{Z} | H_i(X) \neq 0\}$, with the usual convention that $\sup \emptyset = -\infty$ and $\inf \emptyset = \infty$. For an integer ℓ, $\Sigma^\ell X$ is the complex X shifted ℓ degrees to the left. Modules will be considered as complexes concentrated in degree zero and we denote the full subcategory of complexes with homology concentrated in degree zero by $\mathcal{D}_0(R)$. The full subcategory of complexes homologically bounded to the right (resp. left) is denoted by $\mathcal{D}_{\sqsupset}(R)$ (resp. $\mathcal{D}_{\sqsubset}(R)$). Also, the full subcategories of homologically bounded complexes and of complexes with finitely generated homology modules will be denoted by $\mathcal{D}_{\square}(R)$ and $\mathcal{D}^f(R)$, respectively. Throughout for any two properties $\sharp$ and $\natural$ of complexes, we set $\mathcal{D}_{\sharp}^{\natural}(R) := \mathcal{D}_{\sharp}(R) \cap \mathcal{D}^{\natural}(R)$. So for instance, $\mathcal{D}_{\square}^f(R)$ stands for the full subcategory of homologically bounded complexes with finitely generated homology modules.

For any complex X in $\mathcal{D}_{\sqsupset}(R)$ (resp.$\mathcal{D}_{\sqsubset}(R)$), there is a bounded to the right (resp. left) complex P (resp. I) consisting of projective (resp. injective) R-modules which is isomorphic to X in $\mathcal{D}(R)$. A such complex P (resp. I) is called a projective (resp. injective) resolution of X. A complex X is said to have finite projective (resp. injective) dimension, if X possesses a bounded projective (resp. injective) resolution. Similarly, a complex X is said to have finite flat dimension if it is isomorphic (in $\mathcal{D}(R)$) to a bounded complex of flat R-modules. The left derived tensor product functor $-\otimes_R^{\mathbf{L}} \sim$ is computed by taking a projective resolution of the first argument or of the second one. The right derived homomorphism functor $\mathbf{R}\operatorname{Hom}_R(-,\sim)$ is computed by taking a projective resolution of the first argument or by taking an injective resolution of the second one. For any two complexes X and Y and any integer i, set $\operatorname{Ext}_R^i(X,Y) := H_{-i}(\mathbf{R}\operatorname{Hom}_R(X,Y))$. Let X be a complex and $\mathfrak{a}$ an ideal of R. Recall that $\operatorname{Supp}_R X := \cup_{l\in\mathbb{Z}} \operatorname{Supp}_R H_l(X)$, $\operatorname{depth}(\mathfrak{a},X) := -\sup \mathbf{R}\operatorname{Hom}_R(R/\mathfrak{a},X)$ and $\dim_R X := \sup\{\dim R/\mathfrak{p} - \inf X_{\mathfrak{p}} | \mathfrak{p} \in \operatorname{Spec} R\}$. For any complexes X, Y and Z, there are the following natural isomorphisms in $\mathcal{D}(R)$.

Shifts: Let i,j be two integers. Then $\Sigma^i X \otimes_R^{\mathbf{L}} \Sigma^j Y \simeq \Sigma^{j+i}(X \otimes_R^{\mathbf{L}} Y)$ and $\mathbf{R}\operatorname{Hom}_R(\Sigma^i X, \Sigma^j Y) \simeq \Sigma^{j-i}\mathbf{R}\operatorname{Hom}_R(X,Y)$.

Commutativity: $X \otimes_R^{\mathbf{L}} Y \simeq Y \otimes_R^{\mathbf{L}} X$.

Adjointness: Let S be an R-algebra. If $X \in \mathcal{D}_{\sqsupset}(S)$, $Y \in \mathcal{D}(S)$ and $Z \in \mathcal{D}_{\sqsubset}(R)$, then

$$\mathbf{R}\operatorname{Hom}_R(X \otimes_S^{\mathbf{L}} Y, Z) \simeq \mathbf{R}\operatorname{Hom}_S(X, \mathbf{R}\operatorname{Hom}_R(Y,Z)).$$

Tensor evaluation: Assume that $X \in \mathcal{D}_{\sqsupset}^f(R)$, $Y \in \mathcal{D}_{\square}(R)$ and $Z \in \mathcal{D}_{\sqsupset}(R)$. If either projective dimension of X or flat dimension of Z is finite, then

$$\mathbf{R}\operatorname{Hom}_R(X,Y) \otimes_R^{\mathbf{L}} Z \simeq \mathbf{R}\operatorname{Hom}_R(X, Y \otimes_R^{\mathbf{L}} Z).$$

Hom evaluation: Assume that $X \in \mathcal{D}^f_{\sqsupset}(R)$, $Y \in \mathcal{D}_{\square}(R)$ and $Z \in \mathcal{D}_{\sqsubset}(R)$. If either projective dimension of X or injective dimension of Z is finite, then

$$X \otimes^{\mathbf{L}}_R \mathbf{R}\operatorname{Hom}_R(Y,Z) \simeq \mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_R(X,Y),Z).$$

(2.2) Gorenstein homological dimensions. An R-module M is said to be *totally reflexive* if there exists an exact complex P of finitely generated projective R-modules such that $M \cong \operatorname{im}(P_0 \longrightarrow P_{-1})$ and $\operatorname{Hom}_R(P,R)$ is exact. Also, an R-module N is said to be *Gorenstein injective* if there exists an exact complex I of injective R-modules such that $N \cong \operatorname{im}(I_1 \longrightarrow I_0)$ and $\operatorname{Hom}_R(E,I)$ is exact for all injective R-modules E; see [EJ]. Obviously, any finitely generated projective R-module is totally reflexive and any injective R-module is Gorenstein injective. A complex $X \in \mathcal{D}^f_{\square}(R)$ is said to have finite G-dimension if it is isomorphic (in $\mathcal{D}(R)$) to a bounded complex of totally reflexive R-modules. Also, a complex $X \in \mathcal{D}_{\square}(R)$ is said to have finite Gorenstein injective dimension if it is isomorphic (in $\mathcal{D}(R)$) to a bounded complex of Gorenstein injective R-modules.

(2.3) Auslander categories. Let $(R,\mathfrak{m})$ be a local ring. A *normalized dualizing complex* for R is a complex $D \in \mathcal{D}^f_{\square}(R)$ such that the homothety morphism $R \longrightarrow \mathbf{R}\operatorname{Hom}_R(D,D)$ is an isomorphism in $\mathcal{D}(R)$, D has finite injective dimension and $\sup D = \dim R$. Assume that R possesses a normalized dualizing complex D. The Auslander category $\mathcal{A}^f(R)$ (with respect to D) is the full subcategory of $\mathcal{D}^f_{\square}(R)$ whose objects are exactly those complexes $X \in \mathcal{D}^f_{\square}(R)$ for which $D \otimes^{\mathbf{L}}_R X \in \mathcal{D}^f_{\square}(R)$ and the natural morphism $\eta_X : X \longrightarrow \mathbf{R}\operatorname{Hom}_R(D, D \otimes^{\mathbf{L}}_R X)$ is an isomorphism in $\mathcal{D}(R)$. Also, the Auslander category $\mathcal{B}^f(R)$ (with respect to D) is the full subcategory of $\mathcal{D}^f_{\square}(R)$ whose objects are exactly those complexes $X \in \mathcal{D}^f_{\square}(R)$ for which $\mathbf{R}\operatorname{Hom}_R(D,X) \in \mathcal{D}^f_{\square}(R)$ and the natural morphism $\varepsilon_X : D \otimes^{\mathbf{L}}_R \mathbf{R}\operatorname{Hom}_R(D,X) \longrightarrow X$ is an isomorphism in $\mathcal{D}(R)$. By [CFrH, Theorem 4.1 and Prposition 3.8 b)], $\mathcal{A}^f(R)$ precisely consists of all complexes $X \in \mathcal{D}^f_{\square}(R)$ whose G-dimensions are finite. Also, [CFrH, Theorem 4.4] yields that $\mathcal{B}^f(R)$ consists of all complexes $X \in \mathcal{D}^f_{\square}(R)$ whose Gorenstein injective dimensions are finite.

(2.4) Local cohomology. Let $\mathfrak{a}$ be an ideal of R. The right derived functor of $\mathfrak{a}$-section functor $\Gamma_{\mathfrak{a}}(-) = \varinjlim_n \operatorname{Hom}_R(R/\mathfrak{a}^n,-)$ is denoted by $\mathbf{R}\Gamma_{\mathfrak{a}}(-)$. For any complex $X \in \mathcal{D}_{\sqsubset}(R)$, the complex $\mathbf{R}\Gamma_{\mathfrak{a}}(X) \in \mathcal{D}_{\sqsubset}(R)$ is defined by $\mathbf{R}\Gamma_{\mathfrak{a}}(X) := \Gamma_{\mathfrak{a}}(I)$, where I is an (every) injective resolution of X. Also, for any two complexes $X \in \mathcal{D}_{\sqsupset}(R)$ and $Y \in \mathcal{D}_{\sqsubset}(R)$, the generalized $\mathfrak{a}$-section complex $\mathbf{R}\Gamma_{\mathfrak{a}}(X,Y)$ is defined by $\mathbf{R}\Gamma_{\mathfrak{a}}(X,Y) := \mathbf{R}\Gamma_{\mathfrak{a}}(\mathbf{R}\operatorname{Hom}_R(X,Y))$; see [Y]. For any integer i, set $H^i_{\mathfrak{a}}(X,Y) := H_{-i}(\mathbf{R}\Gamma_{\mathfrak{a}}(X,Y))$ and denote $\sup\{i \in \mathbb{Z} | H^i_{\mathfrak{a}}(X,Y) \neq 0\}$ by $\operatorname{cd}_{\mathfrak{a}}(X,Y)$. Let M and N be two R-modules. The notion of generalized local cohomology modules $H^i_{\mathfrak{a}}(M,N) := \varinjlim_n \operatorname{Ext}^i_R(M/\mathfrak{a}^n M, N)$ was introduced by Herzog in his Habilitationsschrift [He]. When M is finitely generated, [Y, Theorem 3.4] yields that $H^i_{\mathfrak{a}}(M,N) \cong H_{-i}(\mathbf{R}\Gamma_{\mathfrak{a}}(M,N))$ for all integers i.

Let $\check{C}(\underline{\mathfrak{a}})$ denote the Čech complex on a set $\underline{\mathfrak{a}} = \{x_1, x_2, \dots, x_n\}$ of generators of $\mathfrak{a}$. So, by the definition, $\check{C}(\underline{\mathfrak{a}}) = \check{C}(x_1) \otimes_R \cdots \otimes_R \check{C}(x_n)$, where for each i, $\check{C}(x_i)$ is the complex $0 \longrightarrow R \longrightarrow R_{x_i} \longrightarrow 0$ concentrated in degrees 0 and -1 in which homomorphisms are the natural ones. For any complex $X \in \mathcal{D}_{\sqsubset}(R)$, [Sc, Theorem 1.1 iv)] implies that $\mathbf{R}\Gamma_{\mathfrak{a}}(X) \simeq X \otimes^{\mathbf{L}}_R \check{C}(\underline{\mathfrak{a}})$.

3. Duality Results

We start by proving two lemmas which are needed in the proof of the main result of this section.

Lemma 3.1. *Let $(R,\mathfrak{m})$ be a local ring possessing a normalized dualizing complex D and $X, Y \in \mathcal{D}^f_{\square}(R)$.*

i) *Assume that one of the following conditions holds:*
a) either projective dimension X or Y is finite,
b) both G-dimension of X and injective dimension of Y are finite.
Then

$$X \otimes_R^{\mathbf{L}} \mathbf{R}\operatorname{Hom}_R(Y, D) \simeq \mathbf{R}\operatorname{Hom}_R(Y, D \otimes_R^{\mathbf{L}} X).$$

ii) *Assume that one of the following conditions holds:*
a) either injective dimension Y or X is finite,
b) both Gorenstein injective dimension of Y and projective dimension of X are finite.
Then

$$X \otimes_R^{\mathbf{L}} \mathbf{R}\operatorname{Hom}_R(Y, D) \simeq \mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_R(D, Y), X).$$

Proof. i) The case a) follows immediately by using commutativity of the bivariant functor $-\otimes_R^{\mathbf{L}} \sim$ and tensor evaluation. Assume that b) holds. Since $Y \in \mathcal{B}^f(R)$, we have $\mathbf{R}\operatorname{Hom}_R(D, Y) \in \mathcal{D}_\square^f(R)$. As Y has finite injective dimension, [C, Theorem 3.3.2 d)] and [C, Theorem A.5.7.2] imply that $\mathbf{R}\operatorname{Hom}_R(D, Y)$ has finite projective dimension. Next, as $Y \in \mathcal{B}^f(R)$ and $\mathbf{R}\operatorname{Hom}_R(D, D) \simeq R$, [C, Lemma 3.3.3 b)] yields that $\mathbf{R}\operatorname{Hom}_R(Y, D) \simeq \mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_R(D, Y), R)$. Now, tensor evaluation, the fact $X \in \mathcal{A}^f(R)$ and [C, Lemma 3.3.3 b)] yield that:

$$\begin{aligned} X \otimes_R^{\mathbf{L}} \mathbf{R}\operatorname{Hom}_R(Y, D) &\simeq X \otimes_R^{\mathbf{L}} \mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_R(D, Y), R) \\ &\simeq \mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_R(D, Y), R) \otimes_R^{\mathbf{L}} X \\ &\simeq \mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_R(D, Y), X) \\ &\simeq \mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_R(D, Y), \mathbf{R}\operatorname{Hom}_R(D, D \otimes_R^{\mathbf{L}} X)) \\ &\simeq \mathbf{R}\operatorname{Hom}_R(Y, D \otimes_R^{\mathbf{L}} X). \end{aligned}$$

ii) Assume that Gorenstein injective dimension of Y is finite. Then $Y \in \mathcal{B}^f(R)$, and so $\mathbf{R}\operatorname{Hom}_R(D, Y) \in \mathcal{D}_\square^f(R)$. As, we saw in i), we have

$$X \otimes_R^{\mathbf{L}} \mathbf{R}\operatorname{Hom}_R(Y, D) \simeq \mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_R(D, Y), R) \otimes_R^{\mathbf{L}} X.$$

As we mentioned above, the finiteness of injective dimension of Y, implies that $\mathbf{R}\operatorname{Hom}_R(D, Y)$ has finite projective dimension. Thus b) and the first case of a) follow by tensor evaluation. It remains to consider the second case of a). So, assume that X has finite injective dimension. Now, as $X \in \mathcal{B}^f(R)$, by using tensor evaluation and Hom evaluation, we can deduce that:

$$\begin{aligned} X \otimes_R^{\mathbf{L}} \mathbf{R}\operatorname{Hom}_R(Y, D) &\simeq \big(D \otimes_R^{\mathbf{L}} \mathbf{R}\operatorname{Hom}_R(D, X)\big) \otimes_R^{\mathbf{L}} \mathbf{R}\operatorname{Hom}_R(Y, D) \\ &\simeq D \otimes_R^{\mathbf{L}} \big(\mathbf{R}\operatorname{Hom}_R(Y, D) \otimes_R^{\mathbf{L}} \mathbf{R}\operatorname{Hom}_R(D, X)\big) \\ &\simeq D \otimes_R^{\mathbf{L}} \mathbf{R}\operatorname{Hom}_R\big(Y, D \otimes_R^{\mathbf{L}} \mathbf{R}\operatorname{Hom}_R(D, X)\big) \\ &\simeq D \otimes_R^{\mathbf{L}} \mathbf{R}\operatorname{Hom}_R(Y, X) \\ &\simeq \mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_R(D, Y), X). \end{aligned}$$

□

The first assertion of the next result was already proved by Foxby [Fo2, Proposition 6.1]. For completeness' sake, we include an easy proof for it. Recall that for a complex $Y \in \mathcal{D}_\sqsubset(R)$, its injective dimension, $\operatorname{id}_R Y$, is defined by

$$\operatorname{id}_R Y := \inf\{\sup\{l \in \mathbb{Z} | I_{-l} \neq 0\} | I \text{ is an injective resolution of Y}\}.$$

Lemma 3.2. *Let $\mathfrak{a}$ be an ideal of R. Let $X \in \mathcal{D}^f_{\sqsupset}(R)$ and $Y \in \mathcal{D}_{\square}(R)$. Then $\mathbf{R}\Gamma_{\mathfrak{a}}(X,Y) \simeq \mathbf{R}\operatorname{Hom}_R(X, \mathbf{R}\Gamma_{\mathfrak{a}}(Y))$. In particular, if X is homologically bounded and not homologically trivial, then $\operatorname{cd}_{\mathfrak{a}}(X,Y) \leq \operatorname{id}_R Y + \sup X$.*

Proof. Let $\underline{\mathfrak{a}}$ be a generating set of $\mathfrak{a}$. As $\check{C}(\underline{\mathfrak{a}})$ is a bounded complex of flat R-modules, tensor evaluation property yields that

$$\mathbf{R}\Gamma_{\mathfrak{a}}(X,Y) \simeq \mathbf{R}\operatorname{Hom}_R(X,Y) \otimes^{\mathbf{L}}_R \check{C}(\underline{\mathfrak{a}}) \simeq \mathbf{R}\operatorname{Hom}_R(X, \mathbf{R}\Gamma_{\mathfrak{a}}(Y)).$$

Now, assume that X is homologically bounded and not homologically trivial. Since $\sup X$ is an integer, we may and do assume that $\operatorname{id}_R Y < \infty$. So, there is a bounded complex I consisting of injective modules such that it is isomorphic to Y in $\mathcal{D}(R)$ and $I_j = 0$ for all $j < -\operatorname{id}_R Y$. One has $\mathbf{R}\Gamma_{\mathfrak{a}}(X,Y) \simeq \mathbf{R}\operatorname{Hom}_R(X, \Gamma_{\mathfrak{a}}(I))$. The complex $\Gamma_{\mathfrak{a}}(I)$ is a bounded complex consisting of injective modules. Now by [C, Corollary A.5.2], we have

$$\operatorname{id}_R Y \geq \operatorname{id}_R \Gamma_{\mathfrak{a}}(I) \geq -\sup X - \inf \mathbf{R}\operatorname{Hom}_R(X, \Gamma_{\mathfrak{a}}(I)).$$

Thus $-\inf \mathbf{R}\Gamma_{\mathfrak{a}}(X,Y) \leq \operatorname{id}_R Y + \sup X$, as claimed. □

Now, we are ready to prove the main result of this section.

Theorem 3.3. *Let $(R,\mathfrak{m})$ be a local ring possessing a normalized dualizing complex D and $X, Y \in \mathcal{D}^f_{\square}(R)$.*

i) *Assume that one of the following conditions holds:*
   a) either projective dimension X or Y is finite,
   b) both G-dimension of X and injective dimension of Y are finite.
   Then
$$\mathbf{R}\Gamma_{\mathfrak{m}}(X,Y) \simeq \operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_R(Y, D \otimes^{\mathbf{L}}_R X), E(R/\mathfrak{m})).$$

ii) *Assume that one of the following conditions holds:*
   a) either injective dimension Y or X is finite,
   b) both Gorenstein injective dimension of Y and projective dimension of X are finite.
   Then
$$\mathbf{R}\Gamma_{\mathfrak{m}}(X,Y) \simeq \operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_R(D,Y),X), E(R/\mathfrak{m})).$$

Proof. Denote the Matlis duality functor $\operatorname{Hom}_R(-, E(R/\mathfrak{m}))$ by $(-)^{\vee}$. By Local Duality Theorem for any complex $Z \in \mathcal{D}^f_{\square}(R)$, we know that $\mathbf{R}\Gamma_{\mathfrak{m}}(Z) \simeq (\mathbf{R}\operatorname{Hom}_R(Z,D))^{\vee}$, see e.g. [Ha, Chapter V, Theorem 6.2]. Using Lemma 3.2 and adjointness yields that:

$$\begin{aligned} \mathbf{R}\Gamma_{\mathfrak{m}}(X,Y) &\simeq \mathbf{R}\operatorname{Hom}_R(X, \mathbf{R}\Gamma_{\mathfrak{m}}(Y)) \\ &\simeq \mathbf{R}\operatorname{Hom}_R(X, \mathbf{R}\operatorname{Hom}_R(Y,D)^{\vee}) \\ &\simeq (X \otimes^{\mathbf{L}}_R \mathbf{R}\operatorname{Hom}_R(Y,D))^{\vee}. \end{aligned}$$

Hence Lemma 3.1 completes the proof. □

Theorem 3.3 has the following immediate corollary.

Corollary 3.4. i) *Let the situation be as in Theorem 3.3 i). Then*

$$-\inf \mathbf{R}\Gamma_{\mathfrak{m}}(X,Y) = \sup \mathbf{R}\operatorname{Hom}_R(Y, D \otimes^{\mathbf{L}}_R X)$$

and

$$-\sup \mathbf{R}\Gamma_{\mathfrak{m}}(X,Y) = \inf \mathbf{R}\operatorname{Hom}_R(Y, D \otimes^{\mathbf{L}}_R X).$$

ii) *Let the situation be as in Theorem 3.3 ii). Then*

$$-\inf \mathbf{R}\Gamma_{\mathfrak{m}}(X,Y) = \sup \mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_R(D,Y),X)$$

and

$$-\sup \mathbf{R}\Gamma_{\mathfrak{m}}(X,Y) = \inf \mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_R(D,Y),X).$$

The first part of the following corollary extends Suzuki's Duality Theorem and its second part extends the Herzog-Zamani Duality Theorem.

Corollary 3.5. *Let $(R,\mathfrak{m})$ be a Cohen Macaulay local ring possessing a canonical module ω. Let M,N be two finitely generated R-modules and i an integer.*

i) *Assume that one of the following conditions holds:*
a) projective dimension of M is finite,
b) both G-dimension of M and injective dimension of N are finite.
Then

$$H^i_{\mathfrak{m}}(M,N) \cong \operatorname{Hom}_R(\operatorname{Ext}_R^{\dim R-i}(N, \omega\otimes_R M), E(R/\mathfrak{m})).$$

ii) *Assume that one of the following conditions holds:*
a) injective dimension of N is finite,
b) both Gorenstein injective dimension of N and projective dimension of M are finite.
Then

$$H^i_{\mathfrak{m}}(M,N) \cong \operatorname{Hom}_R(\operatorname{Ext}_R^{\dim R-i}(\operatorname{Hom}_R(\omega,N),M), E(R/\mathfrak{m})).$$

Proof. i) If G-dimension of M is finite, then by [C, Theorem 3.4.6], one has $\omega\otimes^{\mathbf{L}}_R M \simeq \omega\otimes_R M$. Also, if Gorenstein injective dimension of N is finite, then [C, Theorem 3.4.9] asserts that $\mathbf{R}\operatorname{Hom}_R(\omega,N) \simeq \operatorname{Hom}_R(\omega,N)$. Hence the conclusion is immediate by Theorem 3.3. Note that $\Sigma^{\dim R}\omega$ is a normalized dualizing complex of R. □

Example 3.6. None of Questions 1.1 and 1.2 have positive answers. To see this, let $(R,\mathfrak{m},k)$ be a non-regular Gorenstein local ring. Let $d := \dim R$, and as before, let $(-)^{\vee}$ denote the Matlis duality functor. Since R is Gorenstein, by [C, Theorems 1.4.9 and 6.2.7], both G-dimension of k and Gorenstein injective dimension of k are finite. Assume that one of these questions has an affirmative answer. Then, by Theorem 3.3, it turns out that

$$\operatorname{Ext}^i_R(k,k) \cong \varinjlim_n \operatorname{Ext}^i_R(k/\mathfrak{m}^n k,k) \cong H^i_{\mathfrak{m}}(k,k) \cong \operatorname{Ext}^{d-i}_R(k,k)^{\vee}$$

for all non-negative integers i. This yields that $\operatorname{Ext}^i_R(k,k)=0$ for all $i\notin\{0,1,\dots d\}$. So, R is regular and we get a contradiction.

4. Applications

We start this section by proving a couple of lemmas.

Lemma 4.1. *Let $X \in \mathcal{D}^f_\square(R)$ and $N \in \mathcal{D}^f_0(R)$. Then*

$$-\sup \mathbf{R}\operatorname{Hom}_R(N,X) = \inf\{\operatorname{depth}_{R_\mathfrak{p}} X_\mathfrak{p} | \mathfrak{p} \in \operatorname{Supp}_R N\} = \operatorname{depth}_R(\operatorname{Ann}_R H_0(N), X).$$

Proof. The assertion follows immediately by [Fo2, Proposition 3.4] and [FI, Proposition 2.10]. $\square$

G-dimension of a complex $X \in \mathcal{D}^f_\sqsupset(R)$, $\text{G–dim}_R X$, is defined by

$$\begin{aligned}\text{G–dim}_R X \quad &:= \inf\{\sup\{l \in \mathbb{Z} | Q_l \neq 0\} | Q \text{ is a bounded to the right complex of} \\ &\quad\ \text{totally reflexive R-modules and } Q \simeq X\}.\end{aligned}$$

Lemma 4.2. *Let $(R,\mathfrak{m})$ be a local ring possessing a normalized dualizing complex D, $X \in \mathcal{D}^f_\square(R)$ and M, N two nonzero finitely generated R-modules.*

i) *If M has finite G-dimension and $\operatorname{Supp}_R M \cap \operatorname{Assh}_R N \neq \emptyset$, then*

$$\sup \mathbf{R}\operatorname{Hom}_R(N, D \otimes^{\mathbf{L}}_R M) \geq \dim_R N.$$

ii) *If N has finite Gorenstein injective dimension, then*

$$\sup \mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_R(D,N),X) = \operatorname{depth} R - \operatorname{depth}_R(\operatorname{Ann}_R N, X).$$

Proof. i) Let $\mathfrak{p}$ be a prime ideal of R. From [Fo1, 15.17 c)] and [C, A.8.5.3], one has $\inf D_\mathfrak{p} = \dim R/\mathfrak{p} + \operatorname{depth} R_\mathfrak{p}$. As $M \in \mathcal{A}^f(R)$, by [C, Observation 3.1.7], it follows that $M_\mathfrak{p} \in \mathcal{A}^f(R_\mathfrak{p})$. So, by applying [C, Lemma A.6.4] and [C, A.6.3.2], we can deduce that

$$\begin{aligned}\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \quad &= \operatorname{depth}_{R_\mathfrak{p}}(\mathbf{R}\operatorname{Hom}_{R_\mathfrak{p}}(D_\mathfrak{p}, D_\mathfrak{p} \otimes^{\mathbf{L}}_{R_\mathfrak{p}} M_\mathfrak{p})) \\ &= \operatorname{depth}_{R_\mathfrak{p}}(D_\mathfrak{p} \otimes^{\mathbf{L}}_{R_\mathfrak{p}} M_\mathfrak{p}) + \dim R/\mathfrak{p} + \operatorname{depth} R_\mathfrak{p}.\end{aligned}$$

Thus by Lemma 4.1 and the Auslander-Buchsbaum formula for G-dimension (see e.g. [C, Theorem 1.4.8]), one has:

$$\begin{aligned}\sup \mathbf{R}\operatorname{Hom}_R(N, D \otimes^{\mathbf{L}}_R M) \quad &= -\inf\{-\dim R/\mathfrak{p} - \text{G–dim}_{R_\mathfrak{p}} M_\mathfrak{p} | \mathfrak{p} \in \operatorname{Supp}_R N\} \\ &= \sup\{\dim R/\mathfrak{p} + \text{G–dim}_{R_\mathfrak{p}} M_\mathfrak{p} | \mathfrak{p} \in \operatorname{Supp}_R N\} \\ &= \sup\{\dim R/\mathfrak{p} + \text{G–dim}_{R_\mathfrak{p}} M_\mathfrak{p} | \mathfrak{p} \in \operatorname{Supp}_R M \cap \operatorname{Supp}_R N\} \\ &\geq \dim_R N.\end{aligned}$$

ii) As $N \in \mathcal{B}^f(R)$, one has

$$N \simeq D \otimes^{\mathbf{L}}_R \mathbf{R}\operatorname{Hom}_R(D,N). \quad (*)$$

By [C, A.8.5.3], we have $\inf D = \operatorname{depth} R$. Set $s := \operatorname{depth} R$. Then applying Nakayama's Lemma for complexes (see e.g. [C, Corollary A.4.16]) to $(*)$ yields that

$$\inf \mathbf{R}\operatorname{Hom}_R(D,N) = -\inf D = -s.$$

On the other hand, by [C, Proposition A.4.6], we have

$$\sup \mathbf{R}\operatorname{Hom}_R(D,N) \leq \sup N - \inf D = -s.$$

Hence $\Sigma^s \mathbf{R}\operatorname{Hom}_R(D,N) \in \mathcal{D}_0^f(R)$. From $(*)$, one can conclude that $\Sigma^s \mathbf{R}\operatorname{Hom}_R(D,N)$ and N have the same support, and so Lemma 4.1 implies that

$$\begin{aligned}\sup \mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_R(D,N),X) &= \sup \Sigma^s \mathbf{R}\operatorname{Hom}_R(\Sigma^s \mathbf{R}\operatorname{Hom}_R(D,N),X)\\ &= s + \sup \mathbf{R}\operatorname{Hom}_R(\Sigma^s \mathbf{R}\operatorname{Hom}_R(D,N),X)\\ &= s - \operatorname{depth}_R(\operatorname{Ann}_R H_0(\Sigma^s \mathbf{R}\operatorname{Hom}_R(D,N)),X)\\ &= s - \operatorname{depth}_R(\operatorname{Ann}_R N, X).\end{aligned}$$

$\square$

In the sequel, we establish a characterization of Cohen-Macaulay modules. It partially improves [HZ, Theorem 3.3]. To this end, for a complex $Y \in \mathcal{D}_{\sqsubset}(R)$, we fix the notation $Y^{\perp}$ for the full subcategory of $\mathcal{D}_0^f(R)$ whose objects are exactly those complexes $X \in \mathcal{D}_0^f(R)$ for which $H^i_{\mathfrak{m}}(X,Y) = 0$ for all $i \neq \operatorname{depth}_R Y$.

Proposition 4.3. *Let $(R,\mathfrak{m})$ be a local ring and N a nonzero finitely generated R-module. Consider the following conditions.*

i) *N is Cohen-Macaulay.*
ii) *There is a nonzero R-module $M \in N^{\perp}$ of finite projective dimension such that $\operatorname{Supp}_R M \cap \operatorname{Assh}_R N \neq \emptyset$.*
iii) *There is a nonzero R-module $M \in N^{\perp}$ of finite G-dimension such that $\operatorname{Supp}_R M \cap \operatorname{Assh}_R N \neq \emptyset$.*

Then i) and ii) are equivalent and clearly ii) implies iii). In addition, if either projective or injective dimension of N is finite, then all these conditions are equivalent.

Proof. If a finitely generated R-module M has finite G-dimension, then it is easy to check that the $\widehat{R}$-module $\widehat{M}$ has finite G-dimension too. Also, if a finitely generated R-module M satisfies $\operatorname{Supp}_R M \cap \operatorname{Assh}_R N \neq \emptyset$, then $\operatorname{Supp}_{\widehat{R}} \widehat{M} \cap \operatorname{Assh}_{\widehat{R}} \widehat{N} \neq \emptyset$. So, without loss of generality, we may and do assume that R is complete. So, R possesses a normalized dualizing complex D.

i) $\Rightarrow$ *ii)* Assume that N is Cohen-Macaulay. Then $H^i_{\mathfrak{m}}(R,N) = H^i_{\mathfrak{m}}(N) = 0$ for all $i \neq \operatorname{depth}_R N$, and so $R \in N^{\perp}$.

ii) $\Rightarrow$ *iii)* is clear.

Assume that either projective or injective dimension of N is finite. We show *iii)* implies *i)*. Suppose that there exists a nonzero R-module $M \in N^{\perp}$ which has finite G-dimension. Then $H^i_{\mathfrak{m}}(M,N) = 0$ for all $i \neq \operatorname{depth}_R N$. Hence from Corollary 3.4 i) and Lemma 4.2 i), we deduce that

$$\operatorname{depth}_R N = -\inf \mathbf{R}\Gamma_{\mathfrak{m}}(M,N) = \sup \mathbf{R}\operatorname{Hom}_R(N, D \otimes^{\mathbf{L}}_R M) \geq \dim_R N,$$

and so N is Cohen-Macaulay.

ii) $\Rightarrow$ *i)* is similar to the proof of *iii)* $\Rightarrow$ *i)*. $\square$

Next, we establish the Gorenstein analogue of Proposition 4.3. It is worth to point out that it improves [HZ, Proposition 3.5 1)]. Recall that a non-homologically trivial complex $Y \in \mathcal{D}^f_{\square}(R)$ is said to be Gorenstein if $\operatorname{id}_R Y = \operatorname{depth}_R Y$.

Proposition 4.4. *Let $(R,\mathfrak{m},k)$ be a local ring and $Y \in \mathcal{D}^f_{\square}(R)$ a non-homologically trivial complex. The following are equivalent:*

i) Y *is Gorenstein.*
ii) $Y^{\perp} = \mathcal{D}_0^f(R)$.
iii) $k \in Y^{\perp}$.

Proof. $i) \Rightarrow ii)$ Let $X \in \mathcal{D}_0^f(R)$. By [Y, Theorem 2.7], $\inf\{i \in \mathbb{Z} | H_{\mathfrak{m}}^i(X,Y) \neq 0\} = \operatorname{depth}_R Y$. Since Y is Gorenstein, one has $\operatorname{id}_R Y = \operatorname{depth}_R Y$, and so by Lemma 3.2, it turns out that $H_{\mathfrak{m}}^i(X,Y) = 0$ for all $i \neq \operatorname{depth}_R Y$.

$ii) \Rightarrow iii)$ is clear.

$iii) \Rightarrow i)$ Since $\operatorname{Supp}_R Y \cap \operatorname{Supp}_R k = \{\mathfrak{m}\}$, by [Y, Lemma 2.4], one has $H_{\mathfrak{m}}^i(k,Y) = \operatorname{Ext}_R^i(k,Y)$ for all integers i. Thus $\operatorname{Ext}_R^i(k,Y) = 0$ for all $i \neq \operatorname{depth}_R Y$. By [C, A.5.7.4], this yields that $\operatorname{id}_R Y = \operatorname{depth}_R Y$. □

Lemma 4.5. *Let $(R, \mathfrak{m})$ be a local ring possessing a normalized dualizing complex D and $X \in \mathcal{D}_\square^f(R)$. Assume that X has finite G-dimension. Then*

$$\operatorname{depth}_R(\mathfrak{a}, X) - \dim R \leq \operatorname{depth}_R(\mathfrak{a}, D \otimes_R^{\mathbf{L}} X) \leq \operatorname{depth}_R(\mathfrak{a}, X) - \operatorname{depth} R$$

for all ideals $\mathfrak{a}$ of R.

Proof. Let $\underline{\mathfrak{a}}$ be a generating set for a given ideal $\mathfrak{a}$ of R. As $\check{C}(\underline{\mathfrak{a}})$ is a bounded complex of flat R-modules, $X \in \mathcal{A}^f(R)$ and $\inf D = \operatorname{depth} R$, [CH, Theorem 4.7 i)] and [C, Proposition 3.3.7 a)] imply that

$$\sup \mathbf{R}\Gamma_{\mathfrak{a}}(X) + \operatorname{depth} R \leq \sup \mathbf{R}\Gamma_{\mathfrak{a}}(D \otimes_R^{\mathbf{L}} X) = \sup(D \otimes_R^{\mathbf{L}} \mathbf{R}\Gamma_{\mathfrak{a}}(X)) \leq \sup \mathbf{R}\Gamma_{\mathfrak{a}}(X) + \dim R.$$

But for any complex $Z \in \mathcal{D}_\square^f(R)$, by [Fo2, Proposition 3.14 c)], one has $\operatorname{depth}_R(\mathfrak{a}, Z) = -\sup \mathbf{R}\Gamma_{\mathfrak{a}}(Z)$. This completes the proof. □

The following result can be considered as Grothendieck's non-vanishing Theorem in the context of generalized local cohomology modules. It also improves [DH, Theorem 3.5].

Proposition 4.6. *Let $(R, \mathfrak{m})$ be a local ring, N a nonzero finitely generated R-module and $X \in \mathcal{D}_\square^f(R)$.*

i) *Assume that one of the following conditions is satisfied:*
a) projective dimension of X is finite,
b) G-dimension of X and projective dimension of N are finite.
Then

$$\operatorname{depth} R - \operatorname{depth}(\operatorname{Ann}_R N, X) \leq \operatorname{cd}_{\mathfrak{m}}(X, N) \leq \dim R - \operatorname{depth}(\operatorname{Ann}_R N, X).$$

ii) *Assume that one of the following conditions is satisfied:*
a) injective dimension of N is finite,
b) Gorenstein injective dimension of N and either projective dimension or injective dimension of X are finite.
Then

$$\operatorname{cd}_{\mathfrak{m}}(X, N) = \operatorname{depth} R - \operatorname{depth}(\operatorname{Ann}_R N, X).$$

Proof. Let $Z \in \mathcal{D}_\square^f(R)$. Clearly, then one has $Z \otimes_R \widehat{R} \in \mathcal{D}_\square^f(\widehat{R})$. If projective (resp. injective) dimension of Z is finite, then $Z \otimes_R \widehat{R}$ has finite projective (resp. injective) dimension over $\widehat{R}$. Also, it is easy to check that if G-dimension of Z is finite, then so is G-dimension of $Z \otimes_R \widehat{R}$ over $\widehat{R}$. By [FF, Theorem

3.6] if Gorenstein injective dimension of Z is finite, then so is Gorenstein injective dimension of $Z \otimes_R \widehat{R}$ over $\widehat{R}$. On the other hand, since $\widehat{R}$ is a faithfully flat R-module, for any complex W, one has $\sup W = \sup(W \otimes_R \widehat{R})$ and $\inf W = \inf(W \otimes_R \widehat{R})$. Hence

$$\begin{aligned}
\operatorname{depth}_{\widehat{R}}(\operatorname{Ann}_{\widehat{R}} \widehat{N}, X \otimes_R \widehat{R}) &= \operatorname{depth}_{\widehat{R}}((\operatorname{Ann}_R N)\widehat{R}, X \otimes_R \widehat{R}) \\
&= -\sup \mathbf{R}\operatorname{Hom}_{\widehat{R}}(\widehat{R}/(\operatorname{Ann}_R N)\widehat{R}, X \otimes_R \widehat{R}) \\
&= -\sup(\mathbf{R}\operatorname{Hom}_R(R/\operatorname{Ann}_R N, X) \otimes_R \widehat{R}) \\
&= -\sup \mathbf{R}\operatorname{Hom}_R(R/\operatorname{Ann}_R N, X) \\
&= \operatorname{depth}(\operatorname{Ann}_R N, X).
\end{aligned}$$

Similarly, one has $\operatorname{cd}_{\mathfrak{m}\widehat{R}}(X \otimes_R \widehat{R}, \widehat{N}) = \operatorname{cd}_{\mathfrak{m}}(X, N)$. Thus, we may and do assume that R is complete. Hence R possesses a normalized dualizing complex D. In case i), the G-dimension of X is finite. By using Corollary 3.4 i) and Lemma 4.1, we can deduce that:

$$\operatorname{cd}_{\mathfrak{m}}(X, N) = \sup \mathbf{R}\operatorname{Hom}_R(N, D \otimes^{\mathbf{L}}_R X) = -\operatorname{depth}_R(\operatorname{Ann}_R N, D \otimes^{\mathbf{L}}_R X).$$

Hence Lemma 4.5 completes the proof of i).

In case ii), the Gorenstein injective dimension of N is finite. Hence, the conclusion follows by Corollary 3.4 ii) and Lemma 4.2 ii). □

The following result partially generalizes the Intersection inequality.

Proposition 4.7. *Let $(R, \mathfrak{m})$ be a local ring and M, N two nonzero finitely generated R-modules such that $\operatorname{Supp}_R M \cap \operatorname{Assh}_R N \neq \emptyset$. Assume that one of the following conditions is satisfied:*

i) *projective dimension of M is finite.*
ii) *both G-dimension of M and projective dimension of N are finite.*
iii) *injective dimension of N is finite.*
iv) *both Gorenstein injective dimension of N and injective dimension of M are finite.*

Then

$$\dim_R N \leq \dim_R \mathbf{R}\operatorname{Hom}_R(M, N) \leq -\inf \mathbf{R}\operatorname{Hom}_R(M, N) + \dim_R(M \otimes_R N).$$

Proof. It is easy to check that $\dim_R \mathbf{R}\operatorname{Hom}_R(M, N) = \dim_{\widehat{R}} \mathbf{R}\operatorname{Hom}_{\widehat{R}}(\widehat{M}, \widehat{N})$. Hence, in view of the proof of Proposition 4.6, we may assume that R is complete. So, R possesses a normalized dualizing complex. In each case, it follows that $\mathbf{R}\operatorname{Hom}_R(M, N) \in \mathcal{D}^f_\square(R)$. By Grothendieck's non-vanishing Theorem [Fo2, Proposition 3.14 d)], one has $\operatorname{cd}_{\mathfrak{m}}(M, N) = \dim_R \mathbf{R}\operatorname{Hom}_R(M, N)$. Thus [DH, Corollary 3.2] yields the right hand inequality. In cases i) and ii), the left hand inequality follows by Corollary 3.4 i) and Lemma 4.2 i). Let $\mathfrak{p}_0 \in \operatorname{Supp}_R M \cap \operatorname{Assh}_R N \neq \emptyset$. In each of the cases iii) and iv), our assumptions yield that R is Cohen-Macaulay, and so one has:

$$\begin{aligned}
\dim_R N &= \dim R/\mathfrak{p}_0 \\
&= \dim R - \operatorname{ht} \mathfrak{p}_0 \\
&\leq \dim R - \operatorname{depth}_{R_{\mathfrak{p}_0}} M_{\mathfrak{p}_0} \\
&\leq \dim R - \inf\{\operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} | \mathfrak{p} \in \operatorname{V}(\operatorname{Ann}_R N)\} \\
&= \dim R - \operatorname{depth}_R(\operatorname{Ann}_R N, M).
\end{aligned}$$

Hence in these cases, the left hand inequality follows by Proposition 4.6 ii). □

References

[BS] M. Brodmann and R.Y. Sharp, *Local cohomology: An algebraic introduction with geometric applications*, Cambridge Univ. Press, (1998).

[C] L.W. Christensen, *Gorenstein dimensions*, Lecture Notes in Mathematics, **1747**, Springer-Verlag, Berlin, (2000).

[CH] L.W. Christensen and H. Holm, *Ascent properties of Auslander categories*, Canad. J. Math., **61**(1), (2009), 76-108.

[CFrH] L.W. Christensen, A. Frankild and H. Holm, *On Gorenstein projective, injective and flat dimensions–a functorial description with applications*, J. Algebra, **302**(1), (2006), 231-279.

[DH] K. Divaani-Aazar and A. Hajikarimi, *Generalized local cohomology modules and homological Gorenstein dimensions*, Comm. Algebra, **39**(6), (2011), 2051-2067.

[EJ] E. Enochs and O.M.G. Jenda, *Relative homological algebra*, de Gruyter Expositions in Mathematics, **30**, Walter de Gruyter & Co., Berlin, 2000.

[Fo1] H-B. Foxby, *Hyperhomological algebra & commutative rings*, in preparation.

[Fo2] H-B. Foxby, *Bounded complexes of flat modules*, J. Pure Appl. Algebra, **15**(2), (1979), 149-172.

[FF] H-B. Foxby and A. Frankild, *Cyclic modules of finite Gorenstein injective dimension and Gorenstein rings*, Illinois J. Math., **51**(1), (2007), 67-82.

[FI] H-B. Foxby and S. Iyengar, *Depth and amplitude for unbounded complexes*, Commutative algebra (Grenoble/Lyon, 2001), 119-137, Contemp. Math., **331**, Amer. Math. Soc., Providence, RI, (2003).

[Ha] R. Hartshorne, *Residues and duality*, Lecture Notes in Mathematics, **20**, Springer-Verlag, Berlin-New York, (1966).

[He] J. Herzog, *Komplex Auflösungen und Dualität in der lokalen Algebra*, Habilitationsschrift, Universität Regensburg, (1974).

[HZ] J. Herzog and N. Zamani, *Duality and vanishing of generalized local cohomology*, Arch. Math. (Basel), **81**(5), (2003), 512-519.

[Sc] P. Schenzel, *Proregular sequences, local cohomology, and completion*, Math. Scand., **92**(2), (2003), 161-180.

[Sh] R. Sharp, *Finitely generated modules of finite injective dimension over certain Cohen-Macaulay rings*, Proc. London Math. Soc., **25**(3), (1972), 303-328.

[Su] N. Suzuki, *On the generalized local cohomology and its duality*, J. Math. Kyoto. Univ., **18**(1), (1978), 71-85.

[Y] S. Yassemi, *Generalized section functors*, J. Pure Appl. Algebra, **95**(1), (1994), 103-119.

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